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LETTER TO THE EDITOR

Conformal invariance and line defects in the two-dimensional Ising model

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Abstract. The quantum Ising chain with p equidistant defects is studied. The exact form of the Hamiltonian is found for an infinite number of sites. Using conformal invariance, generalised corner exponents are computed and found to depend continuously on the defect strengths. Realisations of extended conformal algebras in the Hamiltonian spectrum are obtained.

There is increasing interest in the conformal characterisation of two-dimensional statistical systems with moving critical exponents. By investigating the degeneracy pattern of the spectrum of the quantum Hamiltonian, one can establish the appearance of higher dynamical symmetries. This approach has been applied to the Ashkin–Teller quantum chain (von Gehlen and Rittenberg 1986, Baake *et al* 1987a, b) and a D_6-symmetric quantum chain (Schütz 1987). At specific couplings the presence of $N=1$, 2 superconformal invariance and/or Kac–Moody algebras was conjectured. Similar structures occur for operator product algebras with additional higher spin conserved currents (Zamolodchikov 1985, Knizhnik 1986) or for parafermionic algebras (Zamolodchikov and Fateev 1986, Gepner and Qiu 1987).

It is also possible to study models with infinite defect lines. For the Ising model with defect lines, the critical exponents are known (Bariev 1979, McCoy and Perk 1980) to depend continuously on the defect strengths. The critical exponents obtained are generalisations of the standard surface exponents, rather than bulk exponents. In the case of the Ising model with one or two defect lines, one finds a U(1) Kac–Moody algebra as the spectrum-generating algebra, which can be explicitly constructed in terms of fermionic oscillators (Henkel and Patkós 1987a, b).

Here, we are going to study the spectrum of the quantum Ising chain with three defects at the bulk critical point. The Hamiltonian is

$$H=-\tfrac{1}{2}\sum_{n=1}^{N}\sigma^z(n)+\sigma^x(n)\sigma^x(n+1)+\tfrac{1}{2}\sum_{i=1}^{3}(1-\kappa_i)\sigma^x(\tfrac{1}{6}(2i-1)N)\sigma^x(\tfrac{1}{6}(2i-1)N+1) \tag{1}$$

where σ^x and σ^z are Pauli matrices, N is the number of sites of the quantum chain and the defect strengths κ_i $(i=1,2,3)$ are free parameters.

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We shall proceed in three steps. First, using numerical finite-size data, we obtain a conjecture for the exact form of H in the limit $N\to\infty$. We find that H can be written in terms of three free (Dirac) fermions. Second, in the case of p defects, a relationship between the finite-size scaling amplitude of the correlation length and the critical exponent, which in this case is a generalised corner exponent, is obtained. We also find the $N\to\infty$ limit of H and the central charge of the Virasoro algebra for p defects in the Ising model. Finally, using the $N\to\infty$ result for H, we consider the construction of extended conformal algebras.

We now present the numerical part of our investigation. The diagonalisation of H is done in two steps, following a technique of Lieb *et al* (1961). H is written in a fermionic form:

$$\begin{aligned}H=\sum_{n=1}^{N}(C^+(n)C(n)-\tfrac{1}{2})-\tfrac{1}{2}\sum_{n=1}^{N-1}(C^+(n)-C(n))(C^+(n+1)-C(n+1))\\+\tfrac{1}{2}(1-Q)(C^+(N)-C(N))(C^+(1)-C(1))\\+\tfrac{1}{2}\sum_{i=1}^{3}(1-\kappa_i)[C^+(\tfrac{1}{6}(2i-1)N)-C(\tfrac{1}{6}(2i-1)N)]\\\times[C^+(\tfrac{1}{6}(2i-1)N+1)-C(\tfrac{1}{6}(2i-1)N+1)]\end{aligned}\qquad(2)$$

where the $C(n)$ are fermionic operators and Q is the eigenvalue of the operator

$$Q=\tfrac{1}{2}\left[1-\exp\left(\mathrm{i}\pi\sum_{n=1}^{N}C^+(n)C(n)\right)\right].\qquad(3)$$

The spectrum of H is decomposed into the two sectors $Q=0(1)$, which correspond to an even (odd) number of fermionic states.

The diagonalisation of the quadratic form in (2) is done numerically (see also Guimarães and Drugowich de Felicio 1986). The one-fermion energies were computed for a sequence of finite lattices up to $N\approx 200$ sites and the extrapolation towards $N\to\infty$ is done with an algorithm due to Bulirsch and Stoer (1964). A comparison with other frequently used algorithms shows that the Bulirsch–Stoer method yields in general more stable and more reliable results (Henkel and Schütz 1988). The $N\to\infty$ limit of the lowest 20 one-fermion energies can be obtained to at least five digit accuracy. If the energy is measured in units of $6\pi/N$ (this choice of scale will be explained below) we observe the following.

(i) All the higher one-fermion energies can be written as the sum of the energy of one of the lowest six levels and a positive integer.

(ii) The energies of the first and the sixth levels add up to 1 (and also the second and the fifth and the third and the fourth).

We obtain the following form for H (a and b are fermionic operators):

$$\begin{aligned}H=\frac{6\pi}{N}\sum_{i=1}^{3}\left(\sum_{r=0}^{\infty}[(r+\tfrac{1}{2}-\Delta_i)a_r^{+(i)}a_r^{(i)}+(r+\tfrac{1}{2}+\Delta_i)b_r^{+(i)}b_r^{(i)}]-\tfrac{1}{12}(\tfrac{1}{2}-6\Delta_i^2)\right)\\+\varepsilon_{\mathrm{bulk}}N+\varepsilon_{\mathrm{surface}}+\mathrm{O}(N^{-2})\end{aligned}\qquad(4)$$

where $\Delta_i=\Delta_i(\kappa_1,\kappa_2,\kappa_3;Q)$ does depend on all three defect strengths and also on the sector Q. In table 1, some numerical values are given. From these numbers, we observe

$$\begin{aligned}\Delta_1(Q=0)+\Delta_3(Q=1)=\tfrac{1}{2}\\\Delta_2(Q=0)+\Delta_2(Q=1)=\tfrac{1}{2}\\\Delta_3(Q=0)+\Delta_1(Q=1)=\tfrac{1}{2}\end{aligned}\qquad(5)$$

Table 1. Values of $\Delta_i(\kappa_1, \kappa_2, \kappa_3; Q)$ for the quantum Ising chain with three equidistant defects. The expected accuracy is two units in the last given digit.

κ_1	κ_2	κ_3	Δ_1	Δ_2	Δ_3	Q
0.0	0.0	0.5	0.176 208	0.25	0.323 792	0
			0.176 208	0.25	0.323 792	1
0.25	0.25	0.5	0.146 911	0.277 024	0.321 519	0
			0.178 481	0.222 976	0.353 089	1
0.3	0.3	0.3	0.157 227	0.294 377	0.294 377	0
			0.205 623	0.205 623	0.342 774	1
0.3	0.3	0.5	0.137 961	0.285 019	0.320 564	0
			0.179 436	0.214 981	0.362 039	1
0.5	0.5	0.5	0.102 416	0.315 495	0.315 495	0
			0.184 505	0.184 505	0.397 584	1
0.7	0.7	0.5	0.071 018	0.309 168	0.340 567	0
			0.159 434	0.190 832	0.428 982	1
0.7	0.7	0.7	0.055 600	0.327 837	0.327 837	0
			0.172 163	0.172 163	0.444 400	1
0.9	0.9	0.7	0.029 655	0.319 074	0.345 019	0
			0.154 981	0.180 926	0.470 345	1

and if $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$

$$\Delta_1(Q=0) = |\tfrac{1}{4} - (1/\pi)\tan^{-1}(\kappa)| \tag{6}$$

which is the same κ dependence as found in the case of one defect (Henkel and Patkós 1987b). We did not suceed in finding analytic expressions for the other Δ_i.

We note that the convenient normalisation in (4) is $6\pi/N$ rather than the usual $2\pi/N$. This will be explained as follows, combining previous arguments of Turban (1985) and Cardy (1984). Consider a quantum chain with p equidistant defects. By the conformal mapping

$$w = (N/2\pi)\ln u \tag{7}$$

the chain is mapped onto the infinite 2D plane with p half-infinite defect lines emanating from the origin. Now consider the transformation

$$z = u^{\theta/\pi}. \tag{8}$$

If we take $\theta = \pi p$, the 2D plane is mapped onto a Riemannian surface with p sheets, each sheet containing a half-infinite defect line. Combining the conformal mappings (7) and (8), each sheet is mapped onto one section of the strip between two defect lines, of width N/p. For the correlation length we have, repeating the analysis of Turban (1985) and Cardy (1984),

$$\xi_i^{-1} = E_i - E_0 = \frac{2\pi p}{N} x_i(\kappa_1, \ldots, \kappa_p) \tag{9}$$

where the x_i are generalised corner exponents. In the special case $\kappa_1 = \ldots = \kappa_p = 0$, we are back to the corner exponents already studied by Cardy (1984) and Barber *et al* (1984).

For the Ising model, each of the sheets gives rise to one Dirac fermion with a contribution of one to the central charge c of the Virasoro algebra.

However, if p is even, we can alternatively take $\theta = \frac{1}{2}\pi p$ in (8). Consequently, we now have $\frac{1}{2}p$ sheets with an infinite defect line per sheet. Equation (9) remains correct if p is replaced by $\frac{1}{2}p$ and for the central charge c we have in the case of the Ising model

$$c = \begin{cases} p & \text{if } p \text{ is odd} \\ \frac{1}{2}p & \text{if } p \text{ is even.} \end{cases} \tag{10}$$

Our results (9) and (10) are confirmed by our numerical study for $p=3$ discussed above and we also checked it for $p=4$. We note that, for the Ising model, c gives the number of Dirac fermions appearing in the Hamiltonian. To conclude, we have obtained the exact $N \to \infty$ form of the Ising Hamiltonian for p equidistant defects.

Finally, we construct the spectrum in terms of the conformal algebra for the case $p=3$. The generalisation to arbitrary p is obvious.

The scaled energy gaps of the primary fields are in the two sectors $Q = 0, 1$,

$$\frac{N}{6\pi}\Delta E = \sum_{i=1}^{3} [(\Delta_i^2(Q=1) - \Delta_i^2(Q=0))Q/2 + t_i\Delta_i(Q) + \tfrac{1}{2}t_i^2] \tag{11}$$

where the t_i are the eigenvalues of the charge operators

$$\tilde{T}_i = \sum_{r=0}^{\infty} (a_r^{+(i)}a_r^{(i)} - b_r^{+(i)}b_r^{(i)}). \tag{12}$$

The energies of the secondary fields are given by (11) plus a positive integer. By (9), we have thus written the complete list of the critical (corner) exponents. Since the Hamiltonian (4) can be decomposed into three mutually commuting parts, each of which is generated by a U(1) Kac-Moody algebra (Henkel and Patkós 1987b), the partition function $Z = Z_1Z_2Z_3$, where each of the Z_i is the partition function of a quantum Ising chain with one defect. Consequently, the degeneracies of all conformal towers follow the same pattern 1, 3, 5, 22, 51, ..., which are the numbers of partition of the integers into 'tri-coloured' integers. If some of the Δ_i coincide, larger algebras than the $U(1)\otimes U(1)\otimes(1)$ are realised in the model.

We now write the generators of the conformal algebra explicitly. Consider the twisted fermionic fields

$$\begin{aligned} \eta_{j1}(z) &= \sum_{p\in Z+1/2} z^{-p+\Delta_j} a_p^{(j)} \\ \eta_{j2}(z) &= \sum_{p\in Z+1/2} z^{-p-\Delta_j} b_p^{(j)} \end{aligned} \qquad j = 1, 2, 3 \tag{13}$$

where $a_{-p} = a_p^+$ and $b_{-p} = b_p^+$. These fields satisfy the following boundary conditions on the torus:

$$\begin{aligned} \eta_{j1}(\exp(2\pi i)) &= \exp(2\pi i\Delta_j)\eta_{j1}(1) \\ \eta_{j2}(\exp(2\pi i)) &= \exp(-2\pi i\Delta_j)\eta_{j2}(1). \end{aligned} \tag{14}$$

Now, recall that $2n$ (untwisted) Majorana fermions or n Dirac fermions form a realisation of the SO($2n$) Kac-Moody algebra with central charge $k=1$ (Witten 1984, Goddard and Olive 1986). In the same way, we define SO(6) (for $p=3$) Kac-Moody currents from the twisted fermions of (13)

$$T_{ij}{}^{\alpha\beta}(z) = \sum_{n\in Z} T_n z^{n-1} = :\eta_{i\alpha}(z)\eta_{j\beta}(z): + (\Delta_i/z)\delta_{ij}\varepsilon^{\alpha\beta} \tag{15}$$

where $\varepsilon^{\alpha\beta} = 1 - \delta^{\alpha\beta}$ and the dots denote normal ordering. The relationship to the $\tilde{T}_i$ of (12) is

$$\tilde{T}_i = T_{ii}{}^{12}{}_{,0} - \Delta_i. \tag{16}$$

Following the techniques of Sieberg and Schwimmer (1987) one can construct an automorphism onto the untwisted realisation of the SO($2n$) Kac–Moody algebra. The Virasoro field is

$$L(z) = \sum_{n \in Z} L_n z^{-n-2} = \tfrac{1}{2} \sum_{i=1}^{3} \left(\varepsilon^{\alpha\beta} : \frac{\mathrm{d}}{\mathrm{d}z} \eta_{i\alpha}(z) \eta_{i\beta}(z) : + \frac{\Delta_i^2}{z^2} \right) \tag{17}$$

and the Hamiltonian (4) is related to L_0 via

$$L_0 = (6\pi/N)(H + \tfrac{1}{8}). \tag{18}$$

If we have, for example, $\kappa_1 = \kappa_2 = \kappa_3$, two of the Δ_i coincide (see table 1) and we have a realisation of the SU(2)⊗U(1)⊗U(1) Kac–Moody algebra.

Following Schoutens (1987), one may even contemplate non-linear realisations of superconformal algebras in our model. With three Dirac fermions, we can write $N = 4$ superconformal generators

$$G_j^\alpha(z) = \eta_{j\alpha}(z)[(-1)^\alpha T_{33}{}^{12}(z) - i\varepsilon_{jm} T_{mm}{}^{12}(z)] \tag{19}$$

(J, $\alpha = 1, 2$) which indeed close, if the Kac–Moody generators T are taken from the SU(2)⊗SU(2)⊗U(1) subalgebra of SO(6). By the same automorphism already found for the SO(6) Kac–Moody algebra, our twisted realisation of the SO(4)⊗U(1) extended $N = 4$ superconformal algebra can be mapped onto the untwisted form of Schoutens (1987).

Let us summarise our results. For a quantum chain with p equidistant defects, we found a relation (equation (9)) between the energy differences of the Hamiltonian spectrum and generalised corner exponents. For the Ising model, the exact $N \to \infty$ form of the Hamiltonian was obtained and the critical exponents were computed. This Hamiltonian carries realisations of non-Abelian (super-)conformal algebras, which are explicitly given in terms of twisted fermionic fields.

The spectrum of quantum chains with non-equidistant defects is under investigation.

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